1/ Math 112: Introductory Real Analysis

## § Numerical Sequences and Series

Today: Convergent sequences

<u>Def</u> Let $X$ be a metric space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in $X$.

We say that $\{p_n\}$ *converges* to $p \in X$

if for every $\varepsilon > 0$, there is an integer $N$ such that

$$d(p_n, p) < \varepsilon \text{ for any } n \geq N.$$

(That is, for any open neighborhood $U$ of $p$,

$p_n \in U$ for all large enough $n$.)

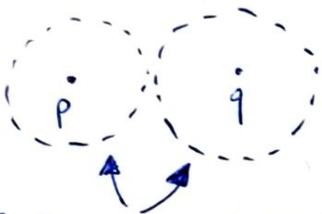
In that case, we say $p$ is the *limit* of $\{p_n\}$

and we write $\lim\limits_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge, it is said to *diverge*.

**Rmk** If $\{p_n\}$ converges, it converges to a unique point.

This is because, if it converges to both $p$ and $q$, with $p \neq q$, then we can take $\varepsilon = \frac{d(p,q)}{2}$ to deduce a contradiction.



$\{p_n\}$ cannot eventually lie in both of these disjoint open balls.

**Rmk** Convergence of $\{p_n\}$ depends not only on $\{p_n\}$ but also on $X$.

For instance, $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ to $0$,

but fails to converge in $\mathbb{R}_{>0}$.

**E.g.** $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{1 + \frac{(-1)^n}{n}\}_{n=1}^{\infty}$ converge in $\mathbb{R}$.

$\{(-1)^n\}_{n=1}^{\infty}$, $\{n^2\}_{n=1}^{\infty}$ diverge in $\mathbb{R}$.

<u>Thm</u> Let $\{p_n\}$ be a sequence in a metric space $X$.

(a) If $\{p_n\}$ converges to $p \in X$ and to $p' \in X$, then $p = p'$.

(b) $\{p_n\}$ converges to $p \in X$ if and only if every open neighborhood of $p$ contains $p_n$ for all but finitely many $n$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and $q$ is a limit point of $E$, then there is a sequence $\{q_n\}$ in $E$ such that $q = \lim_{n\to\infty} q_n$.

Some properties of sequences in $\mathbb{C}$ and $\mathbb{R}^k$:

<u>Thm</u> Suppose $\{s_n\}$ and $\{t_n\}$ are complex sequences such that

$\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.

Then

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$

~~(b) For any $c \in \mathbb{C}$, $\lim_{n\to\infty} c s_n = cs$ and $\lim_{n\to\infty} (c + s_n) = c + s$.~~

(b) $\lim_{n\to\infty} (s_n t_n) = st$

(c) Provided $s_n \neq 0$ for all $n = 1, 2, \cdots$ and $s \neq 0$, $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

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Thm

(a) $\{\mathbf{x}_n\} = \{(\alpha_{1,n}, \cdots, \alpha_{k,n})\} \in \mathbb{R}^k$

converges to $\mathbf{x} = (\alpha_1, \cdots, \alpha_k)$

if and only if $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ for all $1 \leq j \leq k$.

(b) Suppose $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ are sequences in $\mathbb{R}^k$ and $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, $\lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}$ and $\lim_{n\to\infty} \beta_n = \beta$.

Then, $\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}$,

$\lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}$,

$\lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}$.